Enrollment No: ______________________ **Exam Seat No:**______________________

C.U.SHAH UNIVERSITY

Summer Examination-2020

Subject Name : Theories of Ring and Field

Subject Code : 5SC03TRF1 **Branch: M.Sc. (Mathematics)**

Semester : 3 **Date : 03/03/2020** **Time : 02:30 To 05:30** **Marks :70**

Instructions:

(1) Use of Programmable calculator and any other electronic instrument is prohibited.
(2) Instructions written on main answer book are strictly to be obeyed.
(3) Draw neat diagrams and figures (if necessary) at right places.
(4) Assume suitable data if needed.

SECTION – I

Q-1 **Attempt the Following questions** **(07)**

a) Find $[\boldsymbol{Q}(\sqrt{\mathbf{2}}):\boldsymbol{Q}]$. (02)

b) If $\boldsymbol{R}$ is Euclidean ring and $a, b, c \in \boldsymbol{R}$ with $a|b$ and $b|c$ then show that $a|c$. (02)

c) Show that any two GCD of a and b are associate to each other. (02)

d) True/False: If $a, b \in K$, a and b are algebraic over K then ab is algebraic over K (01)

Q-2 **Attempt all questions** **(14)**

a) If π is a prime element in the Euclidean ring $\boldsymbol{R}$ and $\pi|ab$ where $a, b \in \boldsymbol{R}$ then prove that π divides at least one of a and b. (05)

b) Let $\boldsymbol{R}$ be a Euclidean ring with a and $b \neq 0$ in $\boldsymbol{R}$. If b is not a unit in $\boldsymbol{R}$ then prove that $d(a) < d(ab)$. (05)

c) State and prove Wilson's theorem. (04)

OR

Q-2 **Attempt all questions** **(14)**

a) Show that Any two non zero elements a and b in an Euclidean ring $\boldsymbol{R}$ have least common multiple in $\boldsymbol{R}$. (05)

b) Prove that :Any two elements a and b in an Euclidean ring R have greatest common divisor,d.further $d = \lambda a + \mu b$ for some $\lambda, \mu \in R$. (05)

c) If p is a prime integer of the form $4n + 1$, then prove that $p = a^2 + b^2$ for some integers a and b. (04)

Q-3 **Attempt all questions** **(14)**

a) State and prove Eisenstein Criterion. (05)

b) Prove that the given polynomial is irreducible over Q. (05)

$$f(x) = \frac{3}{7}x^4 - \frac{2}{7}x^2 + \frac{9}{35}x + \frac{3}{5}$$

c) Construct a field containing exactly 125 elements. (04)

OR

Q-3		**Attempt all questions**	**(14)**
	a)	If $f(x)$ and $g(x)$ in $\boldsymbol{Z}[x]$ are primitive polynomials then prove that $f(x)g(x)$ is a primitive polynomial.	(05)
	b)	State and prove Gauss's lemma.	(05)
	c)	Let F be a field and $f(x) \in F[x]$ having degree either 2 or 3. Then show that $f(x)$ is irreducible over F if and only if $f(x)$ has a zero in F.	(04)

SECTION – II

Q-4		**Attempt the Following questions**	**(07)**
	a)	Define : Solvable group and give example of it.	(02)
	b)	Define : Radical extension	(02)
	c)	Let $f(x) \in F[x]$ be irreducible. If $char\ F = 0$ then show that $f(x)$ has no multiple root.	(02)
	d)	Write content of the polynomial $x^2 - 2x + 1$.	(01)
Q-5		**Attempt all questions**	**(14)**
	a)	If L is finite extension of K and K is finite extension of F then show that L is finite extension of F. Further show that $[L:F] = [L:K][K:F]$.	(06)
	b)	Find the degree of splitting field of $x^3 - 2$ over $\boldsymbol{Q}$.	(06)
	c)	Prove that $F[x]$ is a integral domain.	(02)

OR

Q-5		**Attempt all questions**	(14)
	a)	If $a \in K$ is algebraic of degree n over F, then show that $[F(a):F] = n$.	(07)
	b)	Prove that A polynomial of degree n over a field can have at most n roots in any extension field.	(07)
Q-6		**Attempt all questions**	**(14)**
	a)	If K is a finite extension of F then $O\big(G(K,F)\big) \leq [K:F] < \infty$	**(05)**
	b)	Any finite extension of characteristic zero is a simple extension.	**(05)**
	c)	Show that $x^3 - 2 \in \boldsymbol{Q}[\boldsymbol{x}]$ is solvable by radicals over $\boldsymbol{Q}$.	**(04)**

OR

Q-6		**Attempt all Questions**	**(14)**
	a)	State and prove Abel's theorem.	**(05)**
	b)	Let $n > 1$ be an integer, F be a field which contains all n^{th} roots of unity and $a \in F\backslash\{0\}$. Let K be the splitting field of $x^n - a \in F[x]$ then show that $K = F(u)$ where $u \in K$ is any root of $x^n - a$.	**(05)**
	c)	Show that the fixed field of G, K_G , is a subfield of K.	**(04)**